

Characteristics of the Wave Function of Coupled Oscillators in Semiquantum Chaos

Gang Wu* and Jinming Dong†

Group of Computational Condensed Matter Physics,
National Laboratory of Solid State Microstructures and Department of Physics,
Nanjing University, Nanjing 210093, P. R. China

(Dated: February 1, 2008)

Using the method of adiabatic invariants and the Born-Oppenheimer approximation, we have successfully got the excited-state wave functions for a pair of coupled oscillators in the so-called *semiquantum chaos*. Some interesting characteristics in the *Fourier spectra* of the wave functions and its *Correlation Functions* in the regular and chaos states have been found, which offers a new way to distinguish the regular and chaotic states in quantum system.

PACS numbers: 05.45. -a, 03.65.Sq

I. INTRODUCTION

Even though plenty of research work has been done on chaos in quantum system, the surprising fact is that its proper definition has not been clearly got now.

From “the principle of quantum corresponding”, there must be counterpart of classical limit for a quantum dynamical system. Percival [1] showed that the semiclassical approximation provides a way to divide the quantum energy spectrum of a N-dimension conservative system into two parts, regular and irregular, which relate, respectively, to periodic and chaotic motions in their classical phase space. Bohigas *et. al.* [2] and Berry *et. al.* [3] then developed the approach of energy-level statistics to describe the properties of regular and irregular quantum energy spectra. Meanwhile, the studies of eigenfunctions made by Shapiro [4] and Heller [5] give us more information about the regular and irregular states of quantum systems.

In quantum mechanics, nonexistence of the concept of path or trajectory prevents us from using the definition of classical chaos, which is characterized by the sensitivity of the system to the initial conditions. There are many theories to study properties of the irregular behavior, or the so-called *quantum chaos*, but most researchers have focused on the concept of integrability in quantum mechanics and hope to get an exact definition of the quantum chaos.

In this paper, we will consider a system in which a classical oscillator interacts with a purely quantum-mechanical oscillator. This system was first considered by Cooper *et.al.* [6], which can be described by a *classical effective Hamiltonian*, the expectation value of the quantum Hamiltonian. Faccioli *et.al.* in their comment [7] gave a more proper treatment for this problem. But Cooper *et.al.* and Faccioli *et.al.* had only paid their attention to the *average value* of the time-dependent occupation number in the chaotic state. As we all know, in quantum mechanics, the wave function can give not only the static but also the dynamical information of the system. Recently, Hui *et. al.* in their paper [8] studied the ground-state wave function of the coupled oscillators, and gave out a method to distinguish the chaotic evolution in

this system. But they did not study the behaviors in its excited state. Then there come some questions: Will this method still work in the excited-state? Could we get also the excited-state wave functions for the coupled oscillators? What are the properties of the excited-state wave functions?

In this paper, we will try to answer these questions, and then to find whether there is another way to reach the goal. In section II, we will use the method of adiabatic invariant [9] and the Born-Oppenheimer Approximation [10-13] to obtain the wave functions in any excited-states. In section III, we will use these wave functions to discuss the time evolution characteristics of the system, and calculate their correlation functions. Finally, we will end the article with some conclusions and discussions in section IV.

II. EXCITED-STATE WAVE FUNCTIONS OF THE COUPLED HARMONIC OSCILLATORS

In this paper, we mainly deal with a coupled system, in which a classical oscillator interacts with a purely quantum mechanical oscillator, described by a classical Lagrangian:

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{A}^2 - \frac{1}{2}(m^2 + e^2 A^2)x^2, \quad (1)$$

where the coordinates x and A describe, respectively, the motion of the quantum oscillator and the classical one in the system. Using the Born-Oppenheimer approximation, we can decouple the system into a classical part and a quantum part, each of which can be handled easily.

From Lagrangian (1), a Schrödinger equation for a state with total energy E_n can be obtained:

$$\frac{1}{2} \left(-\hbar^2 \frac{\partial^2}{\partial x^2} + \omega^2 x^2 - \hbar^2 \frac{\partial^2}{\partial A^2} - 2E_n \right) \Psi_{E_n}(x, A) = 0, \quad (2)$$

where $\omega^2 = m^2 + e^2 A^2$. Factorizing $\Psi_{E_n}(x, A) = \psi(A) \chi_n(x, A)$ and following the treatment in Ref. [7,8],

i.e., using the semiclassical approximation, we obtain the following coupled equations for the $\psi(A)$ and $\chi_n(x, A)$

$$\left(\frac{1}{2}\dot{A}^2 + \langle \hat{H}_x \rangle_n\right) \psi = E_n \psi, \quad (3)$$

and

$$\left(\hat{H}_x - i\hbar \frac{\partial}{\partial t}\right) \chi_{sn} = 0, \quad (4)$$

where

$$\hat{H}_x = \frac{1}{2} \left(-\hbar^2 \frac{\partial^2}{\partial x^2} + \omega^2 \hat{x}^2 \right), \quad (5)$$

$$\chi_n(x, A) = \chi_{sn}(x, A) \exp \left[\int^t dt' \left(\frac{i}{\hbar} \langle \hat{H}_x \rangle_n + \left\langle \frac{\partial}{\partial A'} \right\rangle_n \dot{A}' \right) \right] \quad (6)$$

and for an operator \hat{O} , its average value in n_{th} excited state is defined as

$$\langle \hat{O} \rangle_n = \frac{\int dx \chi_n^* \hat{O} \chi_n}{\int dx \chi_n^* \chi_n}. \quad (7)$$

Now, it is easy to solve the Schrödinger equation, Eq. (4), by using the adiabatic invariant method mentioned in Ref.[1,7,8], in which an adiabatic invariant $I(t)$ and the time-dependent canonical lowering and raising operators \hat{a} and \hat{a}^\dagger are introduced. In order to compare with the paper of Cooper *et.al.*, we let $G = \langle x^2 \rangle_0$. Thus, \hat{a} has the following form:

$$\hat{a} = e^{i\theta} \sqrt{G\hbar} \left[\frac{\partial}{\partial x} + \frac{1-i\dot{G}}{2G\hbar} x \right], \quad (8)$$

and

$$\theta = \int^t dt' \frac{1}{2G}. \quad (9)$$

Further, when we have given the fixed total energy E_n , the A and G satisfy the following equations:

$$\ddot{A} = -\frac{\partial}{\partial A} \langle \hat{H}_x \rangle_n = -(2n+1) \hbar e^2 A G, \quad (10)$$

$$\frac{1}{2} \ddot{G} - \frac{1}{4} \left(\frac{\dot{G}}{G} \right)^2 - \frac{1}{4G^2} + m^2 + e^2 A^2 = 0. \quad (11)$$

On defining the n_{th} excited state of a by $\frac{1}{\sqrt{n!}} (\hat{a})^{n+1} |n\rangle = 0$, we can get in the coordinate representation,

$$\left(\frac{\partial}{\partial x} + \frac{1-i\dot{G}}{2G\hbar} x \right)^{n+1} \phi_n(x, G(t)) = 0, \quad (12)$$

where $\phi_n(x, G(t))$ is just the n_{th} excited state wave function, i.e., $|n\rangle$. Its normalized solution can be easily obtained as follows:

$$|n\rangle = \left(\frac{1}{2G\hbar} \right)^{\frac{1}{4}} H_n \left(\frac{x}{\sqrt{2G\hbar}} \right) \exp \left[-\frac{1}{4G\hbar} (1-i\dot{G}) x^2 \right], \quad (13)$$

where $H_n(x)$ is the n_{th} Hermit function, defined as: $H_n(x) = (-1)^n (2^n n! \sqrt{\pi})^{-\frac{1}{2}} \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2)$. And we take $\hbar = 1$ and $m = 1$ for simplicity.

From Eq. (13), we have

$$|\phi_n(x, G(t))|^2 = \frac{1}{\sqrt{2G}} \left[H_n \left(\frac{x}{\sqrt{2G}} \right) \right]^2 \exp \left(-\frac{x^2}{2G} \right). \quad (14)$$

For a special case of $n = 0$, $\phi_0(x, G(t)) = \left(\frac{1}{2\pi G} \right)^{\frac{1}{4}} \exp \left[-\left(\frac{1-i\dot{G}}{2G} \right) \frac{x^2}{2} \right]$, which is just the ground state wave functions given in Ref. [8].

And it is easy to get the total energy and the quantum part energy:

$$E_n = \frac{1}{2} \dot{A}^2 + (2n+1) \left[\frac{\dot{G}^2 + 1}{8G} + \frac{1}{2} \omega^2 G \right], \quad (15)$$

$$E_{Qn} = (2n+1) \left[\frac{\dot{G}^2 + 1}{8G} + \frac{1}{2} \omega^2 G \right] = E_n - \frac{1}{2} \dot{A}^2 \leq E_n. \quad (16)$$

Especially, when $n = 0$, $E_0 = \frac{1}{2} \dot{A}^2 + \frac{\dot{G}^2 + 1}{8G} + \frac{1}{2} \omega^2 G = \frac{1}{2} \dot{A}^2 + E_{Q0}$.

In order to compare with the results in Ref. [8], we will only fix the total system energy (then, the initial condition and the phase space are fixed.) in the following discussion.

III. DISCUSSIONS ABOUT THE EXCITED-STATE WAVE FUNCTION

Utilizing Eq. (13), we can obtain the *Fourier spectra* (FS) of probability density operator in the excited-state wave functions in chaotic or regular state at the fixed position x . And from them, we can see whether the phenomena happened in ground-state case would be possible

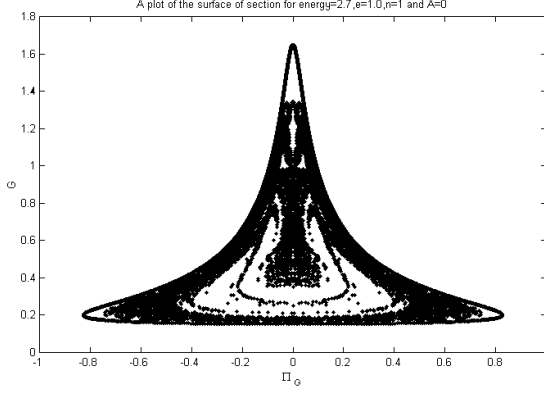


FIG. 1: The Poincaré section in the phase space.

to appear in excited-state. For simplicity, we will use the units of $\hbar = 1$ and $m = 1$, and we take the first excited-state for example. First we show the Poincaré section in the phase space in Fig. 1 by solving the coupled equations (9.5a) and (9.5b).

From Fig. 1, we can easily select the initial conditions as the following:

Chaos State: $A = 0.0, \dot{A} = 1.48565, G = 0.35, \dot{G} = 0.0; e = 1.0;$

Regular State: $A = 0.0, \dot{A} = 1.17969, G = 0.225, \dot{G} = 0.0; e = 1.0;$

And we calculated their *Lyapunov Exponents* (LE) to verify our selection. We find the LE in the first condition is positive and the second one is zero. Fig. 2 shows the FS of the probability density in the first excited state ($n = 1$).

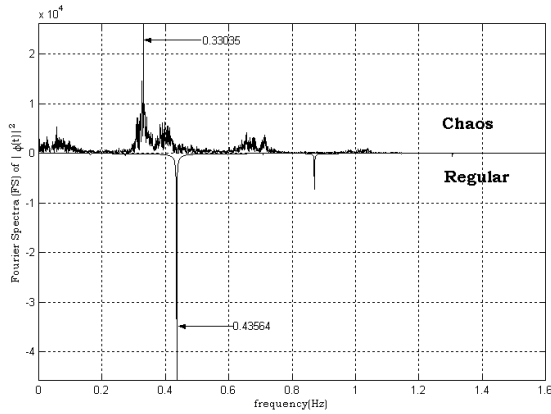


FIG. 2: Fourier spectra of the time evolution of the probability, $|\phi_n(x, G(t))|^2$, where $n=1$.

From Fig. 2., we can see that in the first excited-state, those ‘fundamental frequencies’ still can be found in the Fourier spectra, such as $\nu_1^R=0.4356$ Hz, $\nu_1^C=0.3304$ Hz, etc. They still play important roles in combining the frequencies in the FS.

But if we compare the FS in the first-excited state and

the ground states, we can find much difference between them.

(a). It’s obvious that in the first-excited state the FS in regular state appears more regular than the one in ground state, while the FS of chaotic state appears more chaotic than the one in the ground state.

(b). We can find that in the first-excited state the fundamental frequencies in regular state are larger than the corresponding ones in the ground state, while the fundamental frequencies in chaotic state are smaller than the corresponding ones in the ground state.

In fact, these phenomena can be explained as follows:

(a). It is pointed ([8]) that the ‘fundamental frequencies’ originate only from the coupling between the classical and quantum parts. Because of that reason, the ‘fundamental frequencies’ will also appear in the excited states.

(b). In the first-excited state, the LE of chaotic state ($G(0) = 0.35$) is almost two times larger than that in the ground state ($G(0) = 0.5$). And we find the LE in regular state ($G(0) = 0.225$) converges to zero faster than that in the ground state ($G(0) = 0.35$). So in excited-state, the chaotic state will become more chaotic, and regular state will appear more regular.

From above discussion, we can get a conclusion that the difference between regular and chaotic state in FS still clearly exists in the first excited state. Because of that reason we can use the FS to distinguish the chaotic and regular states even in excited-states, too.

Now, we study the correlation function in the excited states. The *Correlation Function* (CF) is defined as

$$CF(t) = \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^{+L} dt' \int_{-\infty}^{+\infty} dx |\phi(x, t+t')| |\phi(x, t')|, \quad (17)$$

We have got the excited wave function (Eq. 11), so we can get the time-correlation functions in the chaotic and regular state and their FS, respectively.

$$\begin{aligned} CF(t) &= \lim_{L \rightarrow \infty} \frac{1}{L} \int_{-L}^{+L} dt' \int_{-\infty}^{+\infty} dx [4G(t') G(t+t')]^{-\frac{1}{4}} \\ &\quad \times H_n \left(\frac{x}{\sqrt{2G(t')}} \right) H_n \left(\frac{x}{\sqrt{2G(t+t')}} \right) \\ &\quad \times \exp \left\{ -\frac{x^2}{4} \left[\frac{1}{G(t')} + \frac{1}{G(t+t')} \right] \right\} \\ &= \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^{+L} dt' \frac{\sqrt{c}}{2^n n!} \left\{ \left[c_1 \frac{k^n - 1}{k - 1} + c_1^* \frac{(k^*)^n - 1}{(k^*) - 1} \right] + \left[\frac{1}{2} (1 + c^2) \right] \right\} \end{aligned}$$

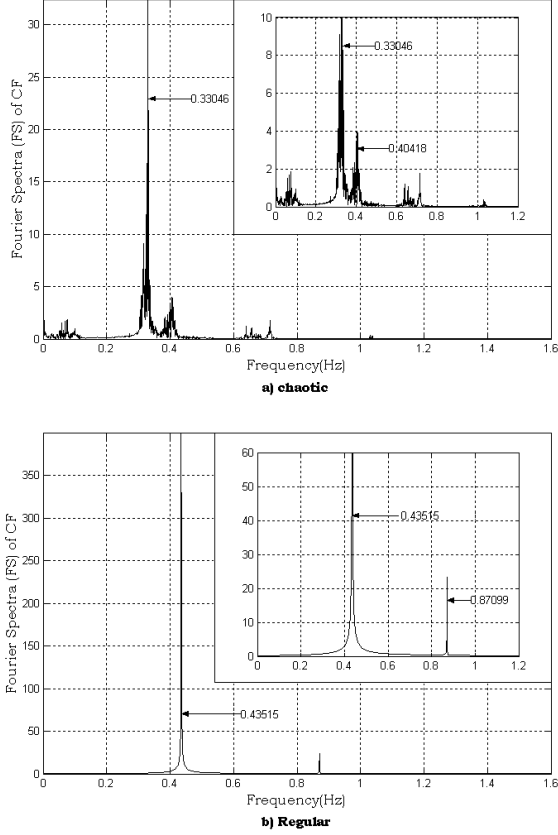


FIG. 3: Fourier spectra of the time-correlation functions of $|\phi_n(x, G(t))|^2$, $n=1$. a) in chaotic state; b) in regular state.

$$= \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^{+L} dt' \frac{\sqrt{c}}{2^n n!} \left\{ 2 \operatorname{Re} \left[c_1 \frac{k^n - 1}{k - 1} \right] + \left[\frac{1}{2} (1 + c^2) \right]^{-\frac{1}{2}} \right\}$$

where

$$c_1 = \alpha_1 + i\beta_1, k = \alpha_2 + i\beta_2,$$

and

$$\alpha_1 = -\frac{1}{2} \frac{c^2 - 4c + 1}{(1 + c^2)^{3/2}},$$

$$\beta_1 = \sqrt{2} \left(\frac{1}{1 + c^2} \right)^{\frac{3}{2}} \frac{c^4 - 6c^2 + 1}{c^2 - 1},$$

$$\alpha_2 = \frac{4c}{1 + c^2}, \beta_2 = \frac{2(c^2 - 1)}{1 + c^2},$$

$$c = \sqrt{\frac{G(t')}{G(t + t')}}.$$

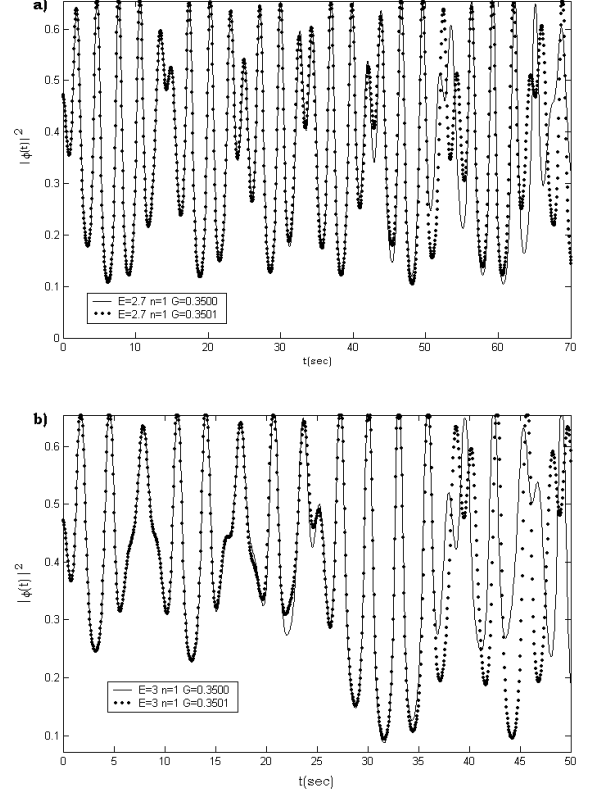


FIG. 4: Sensitivity of the probability density to the initial conditions for different energies. a) $E_1 = 2.7$; b) $E_1 = 3.0$.

In Fig. 3, we give out the Fourier spectra of time-correlation function in the first excited state, in which the mean value has been subtracted from the FS to remove the big dc-component in the FS. In these figures, we can find that those ‘fundamental frequencies’ appear again, and these figures can show the ‘fundamental frequencies’ more clearly. So we know that, the FS of time-correlation can be a justifiable way to distinguish the regular and chaotic states. And when the system is in the chaotic state, the FS of the correlation function becomes desultory, and it looks to have a lot of ‘noise’ components.

Finally, we examine the sensitivity of the eigenfunctions in the chaotic state to the initial conditions, which is usually considered as a basic characteristic of chaos.

Fig.4 shows related numerical results of the probability density $|\phi_n(x, G(t))|^2$ vs. t for the case of $E_1 = 2.7$ (Fig. 4a) and 3.0 (Fig. 4b). The initial conditions are selected by following rule: all parameters are the same except that $G(0)$ has very little difference between two solutions, which are $G(0) = 0.35$ and $G(0) = 0.3501$. We can see that in the first excited state, there still exists such sensitivity for the eigenfunction, which is influenced by the total system energy E_1 . And according to Ref. [8], when the system energy rises, the chaotic behavior becomes stronger.

IV. CONCLUSIONS

In this paper, we analytically calculated the excited-state wave functions of a quantum oscillator coupled with a classical harmonic oscillator. Instead of using the definition of the classical chaos, such as the sensitivity of the system to the initial conditions, we have found the

'*fundamental frequencies*' in the ground state still appear in the excited state, and found a very useful way to get these characteristics. We think these characteristics can be used to distinguish the regular and chaotic states when the so-called *semiquantum chaos* emerges in the actual system and our result is helpful to further research work in this field.

* Electronic address: wugaxp@gmail.com

† Electronic address: jdong@nju.edu.cn

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